Diassociative algebras and their derivations

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Abstract. The paper concerns the derivations of diassociative algebras. We introduce one important class of diassociative algebras, give simple properties of the right and left multiplication operators in diassociative algebras. Then we describe the derivations of complex diassociative algebras in dimension two and three.

1. Introduction

Leibniz algebras and associative dialgebras (dialgebra) first arose in K-theory and rapidly became an object of great interest of many researchers. In 1993, J.Loday introduced the notion of Leibniz algebra [7] which is generalization of Lie algebra, where the skew-symmetricity of the brackets is dropped and the Jacobi identity is replaced by the so-called Leibniz identity. Loday also showed that the link between Lie algebras and associative algebras can be extended to an analogous relationship between Leibniz algebras and the so-called dialgebras (see [8]). Note that the dialgebra is generalization of associative algebra equipped with two products. In particular, it is easy can be shown that if on a vector space V two products $\dashv$ and $\vdash$ are given then the bracket $[\cdot,\cdot]$ by $[x,y] = x \dashv y - x \vdash y$ defines a Leibniz algebra structure on V. Conversely, the enveloping algebra of a Leibniz algebra has the structure of a dialgebra.

In the present paper we deal with the problem of description of derivations of diassociative algebras. The concept of derivation in this case can be easily imitated from that of finite-dimensional algebras. The algebra of derivations plays important role in the classification problems and in different applications of algebras. It is easy to show that the set of all derivations of a diassociative algebra form a Lie algebra with respect to the commutator bracket. In the paper we make use of classification results of two and three-dimensional complex diassociative algebras from [13].

Definition 1.1 *An associative dialgebra (or diassociative algebra) D over a field K is a vector space V over the K equipped with two bilinear associative binary operations denoted by $\dashv$ and $\vdash$, respectively, satisfying the following axioms:*

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \quad (x \vdash y) \dashv z = x \vdash (y \dashv z), \quad (x \dashv y) \vdash z = x \vdash (y \vdash z) \qquad (1)$$

$\forall\ x, y, z \in V$, that is the triple $D = (V, \dashv, \vdash)$ with the axioms above is said to be a diassociative algebra.

Let us consider a few examples of diassociative algebras appeared in the literature.

Example 1.1 *Let $K[x,y]$ be the polynomial algebra over a field K of characteristic* 0. *If we define two multiplications on $K[x,y]$ as follows*

$$f(x,y) \dashv g(x,y) = f(x,y)g(y,y) \quad \text{and} \quad f(x,y) \vdash g(x,y) = f(x,x)g(x,y)$$

then $(K[x,y], \dashv, \vdash)$ is a diassociative algebra.

Example 1.2 *Let $(D, \dashv, \vdash)$ be a diassociative algebra. Consider the module of $n \times n$ - matrices $M_n(D) = M_n(K) \otimes D$ with products $(\alpha \dashv \beta)_{ij} = \Sigma_k \alpha_{ik} \dashv \beta_{kj}$ and $(\alpha \vdash \beta)_{ij} = \Sigma_k \alpha_{ik} \vdash \beta_{kj}$. Then $(M_n(D), \dashv, \vdash)$ is a diassociative algebra. Moreover, if D_1 and D_2 are diassociative algebras over a field K then their tensor product $D_1 \bigotimes_K D_2$ is provided by a dialgebra structure defined as follows:*

$$(a \otimes a') \star (b \otimes b') = (a \star b) \otimes (a' \star b') \quad \text{for} \quad \star = \vdash \text{ and } \dashv .$$

In fact, a diassociative algebra structure on an n-dimensional vector space V with a basis $\{e_1, e_2, ..., e_n\}$ can be given by defining the products of the basis vectors $\{e_1, e_2, ..., e_n\}$.

Example 1.3 *The products*

$$e_1 \vdash e_1 = e_1, \ e_1 \vdash e_2 = e_2, \ e_1 \dashv e_1 = e_1, \ e_2 \dashv e_1 = e_2$$

on two-dimensional and the products

$$e_1 \dashv e_2 = e_1, \ e_2 \dashv e_2 = e_2, \ e_3 \dashv e_3 = e_3, \ e_2 \vdash e_1 = e_1, \ e_2 \vdash e_2 = e_2, \ e_3 \vdash e_3 = e_3$$

on three-dimensional vector spaces define diassociative algebra structures, respectively.

A bar unit in D is an element $e \in D$ such that

$$x \dashv e = x = e \vdash x, \quad \text{for all } x \in D.$$

It is observed that the bar unit is not unique. The set of all bar units of a diassociative algebra is called a halo.

In Example 1.1 any element of the form $1 + (y - x)g(x, y)$ for $g(x, y) \in K[x, y]$ is a bar unit, therefore the halo of the diassociative algebra $K[x, y]$ is the subset $\{1 + (y - x)g(x, y) | g(x, y) \in F[x, y]\}$, meanwhile the identity matrix $I = \begin{pmatrix} 1 & 0 & ... \ 0 \\ 0 & 1 & ... \ 0 \\ ... & ... & ... \\ 0 & 0 & ... \ 1 \end{pmatrix}$ is a bar unit of $M_n(D) = M_n(K) \otimes D$ in Example 1.2 and e_1 is a bar unit in part one of Example 1.3.

Note also that there is no point unit (that is an element $e \in D$ such that $x \vdash e = e \dashv x = x$ for all $x \in D$) in a diassociative algebra, except for the case when the two products coincide, i.e., D is an associative algebra.

Definition 1.2 *A homomorphism of diassociative algebras D_1 and D_2 (provided both are given over the same field K) is a K-linear map $f : D_1 \to D_2$ such that $f(x \vdash y) = f(x) \vdash f(y)$ and $f(x \dashv y) = f(x) \dashv f(y)$ for all $x, y \in D_1$.*

As usual a bijective homomorphism is called isomorphism.

Definition 1.3 *A subspace D_0 of a diassociative algebra D is said to be subalgebra if $x \vdash y$ and $x \dashv y$ are in D_0 whenever $x, y \in D_0$.*

Definition 1.4 *A two-sided ideal of a diassociative algebra D is a subspace I such that $x \star y$, $y \star x$ are in I for all $x \in D$, $y \in I$ with $\star = \vdash$ and $\dashv$. Note that I is called the right and left ideal if $y \vdash x$, $y \dashv x$ are in I, and $x \vdash y$, $x \dashv y$ are in I, respectively, for all $x \in D$, $y \in I$.*

Example 1.4 *Obviously, $I = \{0\}$ and D are two-sided ideals. As well as the kernel $Ker\varphi = \{x \in D_1 | \ \varphi(x) = 0\}$ of a homomorphism $\varphi : D_1 \longrightarrow D_2$ from diassociative algebra D_1 to D_2 is two-sided ideal in D_1 whereas the image $Im\varphi = \{y \in D_2 | \exists x \in D_1 : \ \varphi(x) = y\}$ is just a subalgebra of D_2.*

Let D be a Diassociative algebra and M, N be subsets of D. We define

$$M \lozenge N := M \vdash N + M \dashv N,$$

where

$$M \vdash N = Span_{\mathbb{C}}\{a \vdash b | \ a \in M, b \in N\}$$

and

$$M \dashv N = Span_{\mathbb{C}}\{a \dashv b | \ a \in M, b \in N\}.$$

It is obvious that if M is left (N is right) ideal in D so is $M \lozenge N$, respectively. Therefore, if both M and N are two-sided ideals so is $M \lozenge N$.

Let us consider the following series of two-sided ideals:

$$D^1 = D, \ D^{k+1} = D^1 \lozenge D^k + D^2 \lozenge D^{k-1} + ... + D^k \lozenge D^1 \qquad (2)$$

Definition 1.5 *A Diassociative algebra D is said to be nilpotent if there exists $s \in \mathbb{N}$ such that $D^s = 0$.*

Example 1.5 *Two dimensional algebra with multiplication table $e_1 \vdash e_1 = e_2$, $e_1 \dashv e_1 = \alpha e_2$, $\alpha \in \mathbb{C}$ on a basis $\{e_1, e_2\}$ is a nilpotent diassociative algebra.*

Note that the diassociative algebra in Example 1.2 and in Example 1.3 are not nilpotent.

An ideal I of diassociative algebra D is said to be nilpotent if it is nilpotent as a subalgebra of D.

It is observed that the sum $I_1 + I_2 = \{z \in D | z = x_1 + x_2, \ x_1 \in I_1$ and $x_2 \in I_2\}$ of two nilpotent ideals I_1, I_2 of D is nilpotent. Therefore there exists unique maximal nilpotent ideal of D called nilradical. The nilradical plays an important role in the classification problem of algebras.

Definition 1.6 *A derivation of diassociative algebra D is a linear transformation $d : D \to D$ satisfying*

$$d(x \dashv y) = d(x) \dashv y + x \dashv d(y) \ \text{ and } \ d(x \vdash y) = d(x) \vdash y + x \vdash d(y)$$

for all $x, y \in D$.

The set of all derivations of a diassociative algebra D we denote by $Der(D)$. It is a Lie algebra with respect to the bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$.

Definition 1.7 *A dialgebra D is called characteristically nilpotent if elements of $Der(D)$ are nilpotent with respect to the composition.*

The study of characteristically nilpotent algebras is important in connection with the observations made by Jacobson in [3] and further developments of this concept for different types of algebraic structures we refer to [1], [2], [4], [5], [6], [9], [10], [11] and [12].

2. Results

Since a diassociative algebra possess two binary operations there are two right R_x, r_x and two left L_x, l_x multiplication operators defined as follows

$$R_x(y) := y \dashv x, \quad r_x(y) := y \vdash x,$$
$$L_x(y) := x \dashv y, \quad l_x(y) := x \vdash y.$$

Lemma 2.1 *The sets $R(D) = \{R_x|x \in D\}$, $L(D) = \{L_x|x \in D\}$, $r(D) = \{r_x|x \in D\}$, $l(D) = \{l_x|x \in D\}$ are closed with respect to the composition.*

Proof. The proof can be easily derived from the following identities

$$R_{x \dashv y} = R_y R_x, \qquad L_{x \dashv y} = L_y L_x,$$
$$r_{x \vdash y} = r_y r_x, \qquad l_{x \vdash y} = l_y l_x.$$

The proof of the next lemma also can be easily obtained by simple computations.

Lemma 2.2 *For the right and left multiplication operators of diassociative algebras the following identities hold true:*

$$R_x R_y = R_{r_x(y)}, \qquad R_x r_y = r_{R_x(y)}, \qquad r_x R_y = r_x r_y,$$
$$L_x L_y = L_x l_y, \qquad l_x L_y = L_{l_x(y)}, \qquad l_x l_y = l_{L_x(y)}.$$

Note that the following combinations of the right and left multiplication operators are also derivations of the diassociative algebra D:

$$L_x R_y + L_x L_y \text{ and } l_x r_y + l_x l_y.$$

Definition 2.1 *The subsets $Ann_R(D)$ and $Ann_L(D)$ defined by*

$$Ann_R(D) = \{x \in D|D \dashv x = 0, D \vdash x = 0\}$$

and

$$Ann_L(D) = \{x \in D|x \dashv D = 0, x \vdash D = 0\}$$

of a diassociative algebra D are called the right and the left annihilators of D, respectively.

Lemma 2.3 *The sets $Ann_R(D)$ and $Ann_L(D)$ are two-sided ideals of D.*

Let us consider diassociative algebra $(D, \vdash, \dashv)$ and linear transformations $ad_z(x) = x \dashv z - z \vdash x$. It is not difficult to verify that ad_z is a derivation of D. This type derivations are called inner derivations of diassociative algebra D. The set of all inner derivations we denote by $J(D)$. Then one has

Lemma 2.4 *The subset $J(D)$ is an ideal of the Lie algebra $Der(D)$.*

Proof. Indeed, $ad_{z_1} - ad_{z_2} = ad_{z_1 - z_2}$ and $[d, ad_z] = ad_{d(z)}$, for any $d \in Der(D)$.

Let D be an n-dimensional complex diassociative algebra and $\{e_1, e_2, ..., e_n\}$ be its basis. The components of $e_i \vdash e_j$ and $e_k \dashv e_s$, where $i, j, k, s = 1, 2, ..., n$ on the basis $\{e_1, e_2, ..., e_n\}$ are called the structure constants of D on $\{e_1, e_2, ..., e_n\}$, i.e., if

$$e_i \dashv e_j = \sum_{k=1}^{n} \gamma_{ij}^k e_k, \quad e_i \vdash e_j = \sum_{k=1}^{n} \delta_{ij}^k e_k$$

then the set
$$\{\gamma_{ij}^k, \delta_{st}^q \in K, 1 \leq i, j, k, s, t, q \leq n\}$$
is called the set of structure constants of D. This means that each point $\{\gamma_{ij}^k, \delta_{st}^q\}$ of the affine space K^{2n^3} defines an algebra structure on underlying vector space, however, for this structure to be a diassociative structure the scalars $\{\gamma_{ij}^k, \delta_{st}^q\}$ must satisfy conditions according to the axioms (1) of the diassociative algebra.

Further all the algebras considered are supposed to be over the field of complex numbers $\mathbb{C}$.

Let us now make a discussion on derivations of the diassociative algebras. A derivation d of D we represent in matrix form $d = (d_{ij})_{i,j=1,2,...,n}$ on the basis $\{e_1, e_2, ..., e_n\}$. If the structure constants $\{\gamma_{ij}^k, \delta_{st}^q\}$ are given then we form a system of equations with respect to d_{ij} and solving this system we get the descriptions of the derivations.

This system has the following form:
$$\sum_{k=1}^{n} \gamma_{ij}^k d_{kt} = \sum_{k=1}^{n} (d_{ki}\gamma_{kj}^t + d_{kj}\gamma_{ik}^t), \quad \sum_{k=1}^{n} \delta_{ij}^k d_{kt} = \sum_{k=1}^{n} (d_{ki}\delta_{kj}^t + d_{kj}\delta_{ik}^t), \tag{2}$$
for $1 \leq i, j, t \leq n$.

Let us apply this approach to find the derivations of complex diassociative algebras in dimension two and three. We make use of classification results from [13].

In two dimensional case the system (2) has the following form:
$$\begin{aligned}
d_{12}\gamma_{11}^2 &= d_{21}\gamma_{21}^1 + d_{11}\gamma_{11}^1 + d_{21}\gamma_{12}^1,\\
d_{21}\gamma_{11}^1 + d_{22}\gamma_{11}^2 &= 2d_{11}\gamma_{11}^2 + d_{21}\gamma_{21}^2 + d_{21}\gamma_{12}^2,\\
d_{12}\gamma_{12}^2 &= d_{21}\gamma_{22}^1 + d_{12}\gamma_{11}^1 + d_{22}\gamma_{12}^1,\\
d_{21}\gamma_{12}^1 &= d_{11}\gamma_{12}^2 + d_{21}\gamma_{22}^2 + d_{12}\gamma_{11}^2,\\
d_{12}\gamma_{21}^2 &= d_{12}\gamma_{11}^1 + d_{22}\gamma_{21}^1 + d_{21}\gamma_{22}^1,\\
d_{21}\gamma_{21}^1 &= d_{12}\gamma_{11}^2 + d_{11}\gamma_{21}^2 + d_{21}\gamma_{22}^2,\\
d_{11}\gamma_{22}^1 + d_{12}\gamma_{22}^2 &= d_{12}\gamma_{12}^1 + 2d_{22}\gamma_{22}^1 + d_{12}\gamma_{21}^1,\\
d_{21}\gamma_{22}^1 &= d_{12}\gamma_{12}^2 + d_{22}\gamma_{22}^2 + d_{12}\gamma_{21}^2,
\end{aligned}$$

$$\begin{aligned}
d_{12}\delta_{11}^2 &= d_{21}\delta_{21}^1 + d_{11}\delta_{11}^1 + d_{21}\delta_{12}^1,\\
d_{21}\delta_{11}^1 + d_{22}\delta_{11}^2 &= 2d_{11}\delta_{11}^2 + d_{21}\delta_{21}^2 + d_{21}\delta_{12}^2,\\
d_{12}\delta_{12}^2 &= d_{21}\delta_{22}^1 + d_{12}\delta_{11}^1 + d_{22}\delta_{12}^1,\\
d_{21}\delta_{12}^1 &= d_{11}\delta_{12}^2 + d_{21}\delta_{22}^2 + d_{12}\delta_{11}^2,\\
d_{12}\delta_{21}^2 &= d_{12}\delta_{11}^1 + d_{22}\delta_{21}^1 + d_{21}\delta_{22}^1,\\
d_{21}\delta_{21}^1 &= d_{12}\delta_{11}^2 + d_{11}\delta_{21}^2 + d_{21}\delta_{22}^2,\\
d_{11}\delta_{22}^1 + d_{12}\delta_{22}^2 &= d_{12}\delta_{12}^1 + 2d_{22}\delta_{22}^1 + d_{12}\delta_{21}^1,\\
d_{21}\delta_{22}^1 &= d_{12}\delta_{12}^2 + d_{22}\delta_{22}^2 + d_{12}\delta_{21}^2.
\end{aligned}$$

The possible values of γ_{ij}^k and δ_{st}^q we take from the classification result of [13] mentioned above.

Theorem 2.1 *Any two-dimensional complex diassociative algebra is included in the following isomorphism classes*
$Dias_2^1: e_1 \vdash e_1 = e_1,\ e_1 \dashv e_1 = e_1,\ e_2 \dashv e_1 = e_2;$
$Dias_2^2: e_1 \vdash e_1 = e_1,\ e_1 \vdash e_2 = e_2,\ e_1 \vdash e_1 = e_1;$
$Dias_2^3: e_1 \vdash e_1 = e_2,\ e_1 \dashv e_1 = \alpha e_2,\ \alpha \in \mathbb{C};$
$Dias_2^4: e_1 \vdash e_1 = e_1,\ e_1 \vdash e_2 = e_2,\ e_1 \dashv e_1 = e_1,\ e_2 \dashv e_1 = e_2.$

Let us describe the derivations of $Dias_2^1$. Due to Theorem 2.1 we have $\gamma_{11}^1 = 1$, $\delta_{11}^1 = 1$, $\delta_{21}^2 = 1$ and other $\{\gamma_{ij}^k\ \delta_{st}^q\}$ are zero. Substituting and solving the system above we get the derivations of $Dias_2^1$ as follows

$$d = \begin{pmatrix} 0 & 0 \\ 0 & d_{22} \end{pmatrix}$$

The other cases can be easily found by the same way. As a result one has

Lemma 2.5 *The derivations of two dimensional complex diassociative algebras are given as follows*

Table 1. Derivations of two-dimensional diassociative algebras

Isomorphism Classes	Derivations	Dim. of the derivation algebras
$Dias_2^1$	$\begin{pmatrix} 0 & 0 \\ 0 & d_{22} \end{pmatrix}$	1
$Dias_2^2$	$\begin{pmatrix} 0 & 0 \\ 0 & d_{22} \end{pmatrix}$	1
$Dias_2^3(\alpha),\ \alpha \in \mathbb{C}$	$\begin{pmatrix} d_{11} & 0 \\ d_{21} & 2d_{11} \end{pmatrix}$	2
$Dias_2^4$	$\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$	2

Due to the result of [13] the isomorphism classes of three-dimensional diassociative algebras are given as follows.

Journal of Physics: Conference Series **553** (2014) 012006 doi:10.1088/1742-6596/553/1/012006

Theorem 2.2 *Any three-dimensional complex diassociative algebra is included in the following isomorphism classes of three-dimensional complex diassociative algebras*

$$
\begin{aligned}
Dias_3^1 :& e_1 \dashv e_2 = e_1, e_2 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3, e_2 \vdash e_2 = e_2, e_3 \vdash e_3 = e_3;\\
Dias_3^2 :& e_1 \dashv e_2 = e_1, e_2 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3, e_2 \vdash e_1 = e_1, e_2 \vdash e_2 = e_2, e_3 \vdash e_3 = e_3;\\
Dias_3^3 :& e_1 \dashv e_2 = e_1, e_2 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3, e_2 \vdash e_2 = e_2, e_3 \vdash e_1 = e_1;\\
Dias_3^4 :& e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \vdash e_3 = e_3;\\
Dias_3^5 :& e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \vdash e_1 = e_1 - e_2, e_3 \vdash e_3 = e_3;\\
Dias_3^6 :& e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \vdash e_1 = e_1, e_3 \vdash e_2 = e_2, e_3 \vdash e_3 = e_3;\\
Dias_3^7 :& e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \vdash e_1 = e_2, e_3 \vdash e_2 = e_2, e_3 \vdash e_3 = e_3;\\
Dias_3^8 :& e_1 \dashv e_3 = e_2, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_1 \vdash e_3 = e_2, e_2 \vdash e_3 = e_2,\\
& e_3 \vdash e_1 = e_1 - e_2, e_3 \vdash e_3 = e_3;\\
Dias_3^9 :& e_3 \dashv e_1 = e_2, e_3 \dashv e_2 = e_2, e_3 \dashv e_3 = e_3, e_3 \vdash e_1 = e_1, e_3 \vdash e_2 = e_2, e_3 \vdash e_3 = e_3;\\
Dias_3^{10} :& e_3 \dashv e_1 = e_1, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \vdash e_1 = e_1, e_3 \vdash e_3 = e_3;\\
Dias_3^{11} :& e_3 \dashv e_1 = e_1, e_2 \dashv e_3 = e_2, e_3 \dashv e_3 = e_3, e_3 \vdash e_1 = e_1, e_3 \vdash e_2 = e_2, e_3 \vdash e_3 = e_3;\\
Dias_3^{12} :& e_1 \dashv e_3 = e_1, e_2 \dashv e_3 = e_2, e_3 \dashv e_1 = e_1, e_3 \dashv e_3 = e_3, e_1 \vdash e_3 = e_1, e_3 \vdash e_1 = e_1,\\
& e_3 \vdash e_3 = e_3;\\
Dias_3^{13} :& e_1 \dashv e_3 = e_1, e_2 \dashv e_3 = e_2, e_3 \dashv e_1 = e_1, e_3 \dashv e_3 = e_3, e_1 \vdash e_3 = e_1, e_3 \vdash e_1 = e_1,\\
& e_3 \vdash e_2 = e_2, e_3 \vdash e_3 = e_3;\\
Dias_3^{14} :& e_1 \dashv e_3 = e_1, e_2 \dashv e_3 = e_2, e_3 \dashv e_1 = e_1, e_3 \dashv e_3 = e_3, e_1 \vdash e_3 = e_1 + e_2,\\
& e_3 \vdash e_1 = e_1, e_3 \vdash e_2 = e_2, e_3 \vdash e_3 = e_3;\\
Dias_3^{15} :& e_1 \dashv e_1 = e_2, e_3 \dashv e_3 = e_3, e_3 \vdash e_3 = e_3;\\
Dias_3^{16} :& e_1 \dashv e_3 = e_2, e_3 \dashv e_1 = ke_2, e_1 \vdash e_1 = me_2, e_1 \vdash e_3 = ne_2, e_3 \vdash e_1 = pe_2,\\
& e_3 \vdash e_3 = qe_2;\\
Dias_3^{17} :& e_1 \dashv e_3 = e_2, e_1 \dashv e_2 = e_3, e_2 \dashv e_1 = e_3, e_1 \vdash e_1 = e_2 + e_3, e_1 \vdash e_2 = e_3,\\
& e_2 \vdash e_1 = e_3;
\end{aligned}
$$

where $k, m, n, p, q \in \mathbb{C}$.

Theorem 2.3 *The derivations of three dimensional complex diassociative algebras are given as follows:*

Table 2. Derivations of three-dimensional diassociative algebras

Isom. Class	**Derivations**	**Dim. of Der.alg.**	**Isom. Class**	**Derivations**	**Dim. of Der.alg.**
$Dias_3^1$	$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1	$Dias_3^2$	$\begin{pmatrix} d_{11} & d_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
$Dias_3^3$	$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1	$Dias_3^4$	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	3
$Dias_3^5$	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	3	$Dias_3^6$	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$	4
$Dias_3^7$	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{11} & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$	4	$Dias_3^8$	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$	4
$Dias_3^9$	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$	4	$Dias_3^{10}$,	$\begin{pmatrix} d_{11} & 0 & d_{13} \\ 0 & d_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	3
$Dias_3^{11}$	$\begin{pmatrix} d_{11} & 0 & d_{13} \\ 0 & d_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	3	$Dias_3^{12}$	$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
$Dias_3^{13}$	$\begin{pmatrix} d_{11} & 0 & d_{13} \\ 0 & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$	4	$Dias_3^{14}$	$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$	3
$Dias_3^{15}$	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	3	$Dias_3^{17}$	$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & 3d_{11} & 0 \\ 0 & 0 & 2d_{11} \end{pmatrix}$	1
$Dias_3^{16}$	$\begin{pmatrix} 0 & 0 & 0 \\ d_{21} & 0 & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$	2	$Dias_3^{16}$	$\begin{pmatrix} 0 & 0 & \beta d_{33} \\ d_{21} & 0 & d_{23} \\ 0 & 0 & d_{33} \end{pmatrix}$ $k=0,\ \beta = p - \frac{2q}{n}$	3
	$\begin{pmatrix} 0 & 0 & d_{13} \\ d_{21} & 0 & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$ $n=-p$	3		$\begin{pmatrix} d_{11} & 0 & \alpha d_{11} - 2d_{23} \\ d_{21} & d_{11} & d_{23} \\ 0 & 0 & d_{33} \end{pmatrix}$ $k,m=0\ \alpha = p + \frac{q}{n}$	4
	$\begin{pmatrix} d_{11} & 0 & \alpha d_{11} \\ d_{21} & d_{11} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$ $m=0,\ \alpha = p + \frac{q}{n}$	3		$\begin{pmatrix} d_{11} & 0 & \alpha d_{11} \\ d_{21} & d_{11} & d_{23} \\ \gamma d_{11} & 0 & 0 \end{pmatrix}$ $k=-1, \alpha = p + \frac{q}{n},\ \gamma = p - \frac{m}{n}$	3

where k, m, n, p, q are structure constants of the algebra $Dias_{16}^3$.

As a result of these descriptions we have

Corollary 2.1 *There is no characteristically nilpotent diassociative algebra in dimensions two and three.*

3. Acknowledgement

The research was supported by grants UniKL/IRPS/str11061, Universiti Kuala Lumpur and 05-02-12-2188RU, Universiti Putra Malaysia.

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